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Linear Response and Relaxation in Quantum Lattice Systems

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For spin–lattice systems, the Kubo formula, expressing the relaxation function in terms of the linear response function, is found to be exact in the thermodynamic limit. In addition, analyticity properties are obtained.

KEY WORDS: Quantum lattice; local algebra of observables; linear response function; relaxation function; thermodynamic limit.

1. INTRODUCTION

Thermodynamics when derived from statistical mechanics is concerned with time-dependent correlation functions. The calculation of these functions is in general far too complex. However, one is usually interested in the long-time behavior of correlations between extensive variables, and in this regime the simplifications are considerable.

In the study of time-dependent correlations, the relaxation function plays a very important role. This function describes the manner in which the fluctuations of a system in statistical equilibrium relax. It has been proposed by Kubo[(1)] that the relaxation can be expressed as a time integral of another function which is much more readily calculated—the linear response function. This gives the response of the system to small perturbations.

Several objections have been put forward regarding the validity of Kubo's relation connecting relaxation and linear response (see, e.g., Ref. 2). One of these is the validity of the linear approximation in the perturbation expansion, and another, more serious, perhaps, is the conflict arising from the fact that the perturbation expansion is only valid for short times, while one is interested in the long-time behavior of the relaxation function (see, e.g., Ref. 3).

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In the case of quantum lattice systems with finite-range interactions we show that in the thermodynamic limit the nonlinear contributions to the relaxation function vanish for small times. Moreover, both relaxation function and linear response function are holomorphic for small times, so that one may hope to continue the equation to large times.

The thermodynamic limit is not a restriction. Indeed, for a finite volume a fluctuation of the system will never relax and the relaxation function is almost periodic. Alternatively, one could place the finite system in idealized isothermal surroundings.[5]

A first attempt at the rigorous study of this problem in the thermodynamic limit with quasilocal observables was done in Ref. 4. In the present work we treat extensive observables.

2. DEFINITIONS

Consider the lattice Z^ν, $\nu = 1, 2, \ldots$. For each finite subset Λ of Z^ν let $N(\Lambda)$ denote the number of points in Λ. For simplicity we shall assume invariance under inversion: $-\Lambda \subset \Lambda$.

Let $\mathscr{A}$ denote the quasilocal algebra of the spin–lattice system, i.e., the closure of the union of all local algebras $\mathscr{A}_\Lambda$. Let $\mathscr{A}_L = \cup_\Lambda \mathscr{A}_\Lambda$ be the algebra of local elements. For more details see Ref. 6.

The unperturbed dynamics as usual is given by local Hamiltonians $H_\Lambda \in \mathscr{A}_\Lambda$. These are supposed to be invariant under space translations; one has for all $x \in Z^\nu$

$$\tau_x H_\Lambda = H_{\Lambda + x}$$

$\{\tau_x | x \in Z^\nu\}$ is the group of space translation automorphisms.

In terms of the potential ϕ one has

$$H_\Lambda = \sum_{X \subset \Lambda} \phi(X)$$

where $\phi(X)^* = \phi(X)$, $\phi(X) \in \mathscr{A}_X$, and $\tau_x \phi(X) = \phi(X + x)$. We suppose also that ϕ is of finite range Δ_0; i.e., one has $\phi(X) = 0$ if $0 \in X$ and $X \not\subset \Delta_0$. The norm of the potential is denoted c and is given by

$$c = \sum_X \|\phi(X)\|$$

where the summation is on all subsets X of Z^ν containing the origin of the lattice.

Local time evolution is described by the automorphism α_t^Λ:

$$\alpha^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$$

The uniform limit $\lim_{\Lambda\to\infty} \alpha_t^{\Lambda}(A) = \alpha_t(A)$ exists for all A in $\mathscr{A}$.[7,8] We shall use the following bound, which can be found in Ref. 8: if

$$a_n^{\Lambda} = \underbrace{[H_\Lambda, \dots [H_\Lambda, A] \cdots]}_{n}$$

and $A \in \mathscr{A}_{\tilde{\Lambda}}$, $\tilde{\Lambda} \subset \Lambda$, then

$$\|a_n^{\Lambda}\| \leqslant n!\, \|A\| \{2c \exp[N(\Delta_0)]\}^n \exp[N(\tilde{\Lambda})]$$

Extensive elements are obtained from local elements by a Fourier transform. Let $A \in \mathscr{A}_L$ and $k \in \mathbb{R}^\nu$. Denote

$$A(\Lambda; k) = N(\Lambda)^{-1/2} \sum_{x\in\Lambda} e^{ikx} \tau_x A$$

If A is self-adjoint and Λ has inversion symmetry, then $A(\Lambda; k)$ is self-adjoint again.

Given a self-adjoint local element A, it is used to construct the Hamiltonian of a perturbed dynamics:

$$H_\Lambda^{A;k} = H_\Lambda + A(\Lambda; k)$$

The corresponding automorphisms will be denoted γ_t^{Λ} and are given by

$$\gamma_t^{\Lambda}(B) = [\exp(itH_\Lambda^{A;k})] B \exp(-itH_\Lambda^{A;k})$$

Definition 2.1. Let ω be a state on $\mathscr{A}$. Let A and B be self-adjoint local elements. The relaxation function is given by

$$\Phi_\Lambda{}^{\omega}(A, B; k; t) = \omega((\alpha_{-t}^{\Lambda}\gamma_t^{\Lambda} - 1) B(\Lambda; -k))$$

As has been remarked in the introduction, it can only show relaxational behavior in the thermodynamic limit. The definition differs by a constant from the one found in the literature.[1] This constant is formally given by

$$-\lim_{t\to\infty} \omega((\alpha_{-t}^{\Lambda}\gamma_t^{\Lambda}) B(\Lambda; -k))$$

From Dyson's expansion one has

$$\begin{aligned}
&\gamma_t^{\Lambda} B(\Lambda; -k) \\
&\quad = \alpha_t^{\Lambda} B(\Lambda; -k) + i\int_0^t ds\, [\alpha_s^{\Lambda} A(\Lambda; k), \alpha_t^{\Lambda} B(\Lambda; -k)] - \int_0^t ds \int_0^s ds' \\
&\qquad \times \gamma_{s'}^{\Lambda}([A(\Lambda; k), [\alpha_{s-s'}^{\Lambda} A(\Lambda; k), \alpha_{t-s'}^{\Lambda} B(\Lambda; -k)]])
\end{aligned}$$

Applying the state $\omega\cdot\alpha_{-t}^{\Lambda}$ on this relation,

$$\begin{aligned}
\Phi_\Lambda{}^{\omega}(A, B; k; t) &= +\, i\int_0^t ds\, \omega([\alpha_{-s}^{\Lambda} A(\Lambda; k), B(\Lambda; -k)]) - \int_0^t ds \int_0^s ds' \\
&\qquad \times \omega(\alpha_{-t}^{\Lambda}\gamma_{s'}^{\Lambda}[A(\Lambda; k), [\alpha_{s-s'}^{\Lambda} A(\Lambda; k), \alpha_{t-s'}^{\Lambda} B(\Lambda; -k)]])
\end{aligned}$$

Definition 2.2. The linear response function now is given by

$$\phi_\Lambda{}^\omega(A, B; k; t) = \omega([A(\Lambda; k), \alpha^\Lambda_{-t}B(\Lambda; -k)])$$

The expression for the relaxation function becomes

$$\Phi_\Lambda{}^\omega(A, B; k; t) = -i\int_0^t ds\, \phi_\Lambda{}^\omega(B, A; -k; s) + \cdots$$

Compare this to the Kubo formula, which in our notations reads

$$\Phi_\Lambda{}^\omega(A, B; k; t) = \lim_{\varepsilon\to 0^+} i\int_t^\infty ds\, \phi_\Lambda{}^\omega(B, A; -k; s)e^{-\varepsilon s}$$

For applications of the situation described in this paper, see, e.g., Ref. 9.

3. BOUND ON THE NONLINEAR TERM

Lemma 3.1. Let A and B be local elements. There exist $t_0 > 0$ and $M > 0$ such that for all s and t in $[-t_0, t_0]$, for all k and k' in R^ν, and for all finite subsets Λ of Z^ν one has

$$\|[A(\Lambda; k), [\alpha_s{}^\Lambda A(\Lambda; k), \alpha_t{}^\Lambda B(\Lambda; k')]]\| < N(\Lambda)^{-1/2}M$$

Proof

One has

$$\begin{aligned}
&[A(\Lambda; k), [\alpha_s{}^\Lambda A(\Lambda; k), \alpha_t{}^\Lambda B(\Lambda; k')]] \\
&\quad = N(\Lambda)^{-3/2}\sum_{x,y,z\in\Lambda} e^{ik(x+y)}e^{ik'z}[\tau_x A, [\alpha_s{}^\Lambda\tau_y A, \alpha_t{}^\Lambda\tau_z B]] \\
&\quad = N(\Lambda)^{-3/2}\sum_{x,y,z\in\Lambda} e^{ik(x+y)}e^{ik'z}\sum_{n,m=0}^\infty \frac{(is)^n}{n!}\frac{(it)^m}{m!}c_{nm}(x, y, z) \qquad (1)
\end{aligned}$$

with

$$c_{nm}(x, y, z) = [\tau_x A, [H_\Lambda,\ldots [H_\Lambda, \tau_y A]\underbrace{\cdots]}_{n}, [H_\Lambda,\ldots [H_\Lambda, \tau_z B]\underbrace{\cdots]]}_{m}$$

Let $A \in \mathscr{A}_{\Delta_A}$ and $B \in \mathscr{A}_{\Delta_B}$. Then $[H_\Lambda,\ldots [H_\Lambda, \tau_y A]\underset{n}{\cdots}]$ belongs to $\mathscr{A}_{\Delta_A+y+n\Delta_0}$. Similarly, one has $[H_\Lambda,\ldots [H_\Lambda, \tau_z B]\underset{m}{\cdots}] \in \mathscr{A}_{\Delta_B+z+m\Delta_0}$. The commutator of both expressions therefore is zero if the intersection

$$(\Delta_A + y + n\Delta_0) \cap (\Delta_B + z + m\Delta_0) \text{ is empty} \qquad (2)$$

In any case this commutator belongs to $\mathscr{A}_{(\Delta_A+y+n\Delta_0)\cup(\Delta_B+z+m\Delta_0)}$. In the expression for $c_{nm}(x, y, z)$ it is commuted again with $\tau_x A$. For a nonvanishing result one needs therefore

$$(\Delta_A + x) \cap [(\Delta_A + y + n\Delta_0) \cup (\Delta_B + z + m\Delta_0)] \neq \varnothing \qquad (3)$$

The previous arguments can be used to obtain a bound on the number of points y and z for which $c_{nm}(x, y, z)$ does not vanish. The point x is kept fixed.

From (3) one sees that either $y \in 2\Delta_A + n\Delta_0 + x$ or $z \in \Delta_A + \Delta_B + m\Delta + x$. Take e.g., $y \in 2\Delta_A + n\Delta_0 + x$. From (2) there follows $z \in \Delta_A + \Delta_B + y + (n + m)\Delta_0$ and hence $z \in 3\Delta_A + \Delta_B + x + (2n + m)\Delta_0$. In the other case ($z \in \Delta_A + \Delta_B + m\Delta_0 + x$) there follows $y \in \Delta_A + \Delta_B + z + (n + m)\Delta_0$ and hence $y \in 2\Delta_A + 2\Delta_B + (n + 2m)\Delta_0 + x$. Let $\Gamma = 3\Delta_A + 2\Delta_B + 2\Delta_0$. Then in any case y and z belong to $(n + m)\Gamma + x$. Therefore each of the y and z summations contain at most $(n + m)^\nu N(\Gamma)$ terms.

Let us now look for a bound in norm on $c_{nm}(x, y, z)$. One has

$$\|c_{nm}(x, y, z)\| \leqslant 4\|A\| \, \|[H_\Lambda, \ldots [H_\Lambda, \tau_y A] \underbrace{\cdots]]}_{n}\| \, \|[H_\Lambda, \ldots [H_\Lambda, \tau_z B] \underbrace{\cdots]]}_{m}\|$$

With the bound mentioned in the previous section this becomes

$$\leqslant 4\|A\|^2\|B\| \, n! \, m! \, [2c \exp N(\Delta_0)]^{n+m} \exp[N(\Delta_A) + N(\Delta_B)]$$

The two results obtained so far together yield

$$\begin{aligned}&\Big\| \sum_{x,y,z\in\Lambda} e^{ik(x+y)} e^{ik'z} c_{nm}(x, y, z) \Big\| \\ &\quad \leqslant 4\|A\|^2\|B\| \, n! \, m! \, [2c \exp N(\Delta)]^{n+m} \exp[N(\Delta_A) + N(\Delta_B)] \\ &\quad\quad \times N(\Lambda)(n + m)^{2\nu} N(\Gamma)^2\end{aligned}$$

The bound for (1) becomes

$$\begin{aligned}&N(\Lambda)^{-1/2} 4\|A\|^2\|B\| \exp[N(\Delta_A) + N(\Delta_B)] \, N(\Gamma)^2 \\ &\quad \times \sum_{n,m=0}^{\infty} |s|^n |t|^m [2c \exp N(\Delta_0)]^{n+m} (n + m)^{2\nu}\end{aligned}$$

The series is uniformly bounded for s and t in $[-t_0, t_0]$ if t_0 satisfies

$$0 < t_0 < (2c)^{-1} \exp[-N(\Delta_0)]$$

Hence the lemma follows. ■

Theorem 3.2. Let A and B be self-adjoint local elements of the algebra of the quantum spin lattice. Let the time evolution be determined by a finite-range interaction as explained above. Then there exist $t_0 > 0$ and $M > 0$ such that

$$\left| \Phi_\Lambda{}^\omega(A, B; k; t) + i\int_0^t ds \, \phi_\Lambda{}^\omega(B, A; -k; s) \right| < N(\Lambda)^{-1/2} M$$

for all t in $[-t_0, t_0]$, for all $k \in R^\nu$, for all states ω on $\mathscr{A}$, and for all finite subsets Λ of Z^ν that are inversion invariant.

Hence

$$\lim_{\Lambda\to\infty}\left|\Phi_\Lambda{}^\omega(A, B; k; t) - \omega(B(\Lambda; -k)) + i\int_0^t ds\, \phi_\Lambda{}^\omega(B, A; -k; s)\right| = 0$$

Proof. From the previous section one has

$$\begin{aligned}&\Phi_\Lambda{}^\omega(A, B; k; t)\\ &\quad -i\int_0^t ds\, \phi_\Lambda{}^\omega(B, A; -k; s)\\ &\quad -\int_0^t ds\int_0^s ds'\, \omega(\alpha^\Lambda_{-t}\gamma^\Lambda_{s'}([A(\Lambda; k), [\alpha^\Lambda_{s-s'}A(\Lambda; k), \alpha^\Lambda_{t-s'}B(\Lambda; -k)]]))\end{aligned}$$

The nonlinear term is majorized by

$$\tfrac{1}{2}t^2\|[A(\Lambda; k), [\alpha^\Lambda_{s-s'}A(\Lambda; k), \alpha^\Lambda_{t-s'}B(\Lambda; -k)]]\|$$

From the lemma one gets $t_0' > 0$ and $M' > 0$ such that if $|s - s'| < t_0'$ and $|t - s'| < t_0'$ the bound becomes $\frac{1}{2}t^2N(\Lambda)^{-1/2}M'$. Let $t_0 = \frac{1}{2}t_0'$ and $M = \frac{1}{2}(t_0')^2M'$. The theorem follows. ∎

4. THERMODYNAMIC LIMIT

In this section it is shown that the linear response function exists in the thermodynamic limit and some properties are derived. For local elements A and B this is straightforward. Indeed, the linear response function $\omega([B, \alpha^\Lambda_{-t}A])$ converges to $\omega([B, \alpha_{-t}A])$. For extensive observables $A(\Lambda; k)$ and $B(\Lambda; -k)$ one has to study the limit

$$\lim_{\Lambda\to\infty}\omega([A(\Lambda; k), \alpha^\Lambda_{-t}B(\Lambda; -k)])$$

In the following A and B are assumed to be self-adjoint local elements, and we denote

$$C^\Lambda_{m,m'}(k, t) = \sum_{n=m}^{m'}\left(\frac{-it}{n!}\right)^n\sum_{x\in\Lambda}e^{ikx}[\tau_xA, \underbrace{[H_\Lambda, \ldots [H_\Lambda, B]\cdots]]}_{n}$$

Lemma 4.1. Let $k_0 > 0$. There exists $t_0 > 0$ such that for $|t| \leqslant t_0$ and $|\mathrm{Im}\, k_\alpha| \leqslant k_0$ the limit

$$\lim_{\Lambda\to\infty}\sum_{x\in\Lambda}e^{ikx}[\tau_xA, \alpha^\Lambda_{-t}B]$$

converges in norm to an element of $\mathscr{A}$, which will be denoted $\xi(A, B; k; t)$. The convergence is uniform in k and t. The norm $\|\xi(A, B; k; t)\|$ is uniformly bounded in k and t.

Proof. Because $[H_\Lambda, \ldots \underbrace{[H_\Lambda, B] \cdots]}_{n}$ belongs to $\mathscr{A}_{\Delta_B + n\Delta}$ the commutator of this expression with $\tau_x A$ vanishes if $(\Delta_A + x) \cap (\Delta_B + n\Delta_0) = \varnothing$. Hence only terms with $x \in \Delta_A + \Delta_B + n\Delta_0$ can contribute to the sum. The number of x's is majorized by $n^\nu N(-\Delta_A + \Delta_B + \Delta_0)$.

Denote $f_n = \max\{|e^{ikx}|$, with $|\mathrm{Im}\, k_\alpha| \leqslant k_0$ and $x \in -\Delta_A + \Delta_B + n\Delta_0\}$. Then one has $f_n \leqslant f_0 g^n$, with $g = \max\{|e^{ikx}|$, with $|\mathrm{Im}\, k_\alpha| \leqslant k_0$ and $x \in \Delta_0\}$. We also have the bound

$$\|[\tau_x, A, [H_\Lambda, \ldots \underbrace{[H_\Lambda, B] \cdots]]}_{n}\| \leqslant 2\|A\| \, \|B\| \, n! \, [2c \exp N(\Delta)]^n \exp N(\Delta_B)$$

There follows

$$\|C^\Lambda_{m,m'}(k, t)\| \leqslant \sum_{n=m}^{m'} |t|^n f_0 g^n n^\nu N(-\Delta_A + \Delta_B + \Delta_0) \times 2\|A\| \, \|B\| [2c \exp N(\Delta)]^n \exp N(\Delta_B)$$

Let t_0 be such that $0 < t_0 < [2gc \exp N(\Delta_0)]^{-1}$. Then the series

$$\sum_{n=0}^{\infty} |t|^n g^n n^\nu [2c \exp N(\Delta_0)]^n$$

is holomorphic on the disk $|t| \leqslant t_0$. We then conclude that the series $C^\Lambda_{0,\infty}(k, t)$ converges in norm to $\sum_{x \in \Lambda} e^{ikx} [\tau_x A, \alpha^\Lambda_{-t} B]$ uniformly in t, k, and Λ and that the latter expression is uniformly bounded in t, k, and Λ.

Remark that if Λ is large enough so as to include $+\Delta_A + \Delta_B + m'\Delta_0$, then $C^\Lambda_{0,m'}(k, t)$ no longer depends on Λ and will be denoted by $C_{0,m'}(k, t)$. From the convergence of $C^\Lambda_{0,\infty}(k, t)$ uniformly in Λ then follows the convergence of $\lim_{m' \to \infty} C_{0,m'}(k, t)$ to some element of $\mathscr{A}$, which will be denoted $\xi(A, B; k; t)$. The latter convergence is also uniform in k and t and $\|\xi(A, B; k; t)\|$ is uniformly bounded in k and t.

Let $\epsilon > 0$. There exists an index m such that

$$\|\xi(A, B; k; t) - C_{0,m}(k, t)\| < \epsilon$$

and

$$\|C^\Lambda_{0,m}(k, t) - C_{0,\infty}(k, t)\| < \epsilon$$

for all k, t, and Λ. But for Λ large enough one has $C^\Lambda_{0,m}(k, t) = C_{0,m}(k, t)$. Hence

$$\|\xi(A, B; k; t) - C^\Lambda_{0,\infty}(k, t)\| < \epsilon$$

This shows the convergence of $\lim_{\Lambda \to \infty} C^\Lambda_{0,\infty}(k, t)$ to $\xi(A, B; k; t)$ uniformly in k and t. ■

Corollary 4.2. For any state ω on $\mathscr{A}$ the function $\omega(\xi(A, B; k; t))$ is holomorphic in k and t on the domains specified in the lemma. This follows because $\omega(\xi(A, B; k; t))$ is a uniformly bounded limit of holomorphic functions.

Lemma 4.3. Let $t_0 > 0$. The limit

$$\lim_{\Lambda\to\infty} \frac{1}{N(\Lambda)} \sum_{y\in\Lambda} [C^{\Lambda}_{m,m'}(k, t) - C^{\Lambda-y}_{m,m'}(k, t)]$$

converges to zero uniformly in $k \in R^{\nu}$ and $t \in [-t_0, t_0]$.

Proof. One has

$$\begin{aligned}
&C^{\Lambda}_{m,m'}(k, t) - C^{\Lambda-y}_{m,m'}(k, t)\\
&= \sum_{n=m}^{m'} \left(\frac{-it}{n!}\right)^n \Big\{ \sum_{x\in\Lambda} e^{ikx}[\tau_x A, \underbrace{[H_\Lambda, \dots [H_\Lambda, B] \cdots]]}_{n}\\
&\quad - \sum_{x\in\Lambda-y} e^{ikx}[\tau_x A, \underbrace{[H_{\Lambda-y}, \dots [H_{\Lambda-y}, B] \cdots]]}_{n} \Big\}\\
&= \sum_{n=m}^{m'} \left(\frac{-it}{n!}\right)^n \Big\{ \sum_{x\in\Lambda\cap(\Lambda-y)} e^{ikx}[\tau_x A, \{\underbrace{[H_\Lambda, \dots [H_\Lambda, B] \cdots]}_{n}\\
&\quad - \underbrace{[H_{\Lambda-y}, \dots [H_{\Lambda-y}, B] \cdots]}_{n}\}] + \sum_{x\in\Lambda\backslash(\Lambda-y)} e^{ikx}[\tau_x A, \underbrace{[H_\Lambda, \dots [H_\Lambda, B] \cdots]]}_{n}\\
&\quad - \sum_{x\in(\Lambda-y)\backslash\Lambda} e^{ikx}[\tau_x A, \underbrace{[H_{\Lambda-y}, \dots [H_{\Lambda-y}, B] \cdots]]}_{n} \Big\}
\end{aligned}$$

But one has

$$\begin{aligned}
&[\tau_x A, \{\underbrace{[H_\Lambda, \dots [H_\Lambda, B] \cdots]}_{n} - \underbrace{[H_{\Lambda-y}, \dots [H_{\Lambda-y}, B] \cdots]}_{n}\}]\\
&\quad = \left(\sum_{Z_1,\dots,Z_n\subset\Lambda} - \sum_{Z_1,\dots,Z_n\subset\Lambda-y} \right) [\tau_x A, \underbrace{[\phi(Z_1), \dots [\phi(Z_n), B] \cdots]]}_{n}
\end{aligned}$$

Terms for which Z_1 through Z_n lie in $\Lambda \cap (\Lambda - y)$ two by two cancel. Suppose now that Λ is large enough so that $\Delta_B + n\Delta_0 \subset \Lambda$. Then $Z_i \subset \Lambda$ follows. Hence in the foregoing expression only the first summation remains, with the restriction that at least one of the Z_i has nonzero intersection with the complement $C_{\Lambda-y}$ of $\Lambda - y$. We conclude that foregoing expression vanishes if y does not belong to the set $C_\Lambda + n\Delta_0$.

Next consider the sum

$$\sum_{x\in\Lambda\backslash(\Lambda-y)} e^{ikx}[\tau_x A, \underbrace{[H_\Lambda, \dots [H_\Lambda, B] \cdots]]}_{n}$$

The commutators vanish if not $x \in \Delta_A + \Delta_B + n\Delta_0$. From $x \notin \Lambda - y$ it then follows that $y \in C_\Lambda + \Delta_A + \Delta_B + n\Delta_0$. The sum

$$\sum_{x\in(\Lambda-y)\backslash\Lambda} e^{ikx}[\tau_x A, \underbrace{[H_{\Lambda-y}, \dots [H_{\Lambda-y}, B] \cdots]]}_{n}$$

vanishes if Λ is taken large enough so as to contain the set $\Delta_A + \Delta_B + n\Delta_0$.

We conclude that $C^{\Lambda}_{m,m'}(k, t) - C^{\Lambda-y}_{m,m'}(k, t)$ is zero if y does not belong to $C_\Lambda + \Delta_A + \Delta_B + m'\Delta_0$.

On the other hand, a uniform bound M for $\|C^{\Lambda}_{m,m'}(k, t) - C^{\Lambda-y}_{m,m'}(k, t)\|$ can be obtained by the methods of the previous lemma. There follows

$$\left\| \frac{1}{N(\Lambda)} \sum_{y\in\Lambda} [C^{\Lambda}_{m,m'}(k, t) - C^{\Lambda-y}_{m,m'}(k, t)] \right\| \leqslant M \frac{N(\Lambda \cap (C_\Lambda + \Delta_A + \Delta_B + m'\Delta_0))}{N(\Lambda)}$$

The set $\Lambda \cap (C_\Lambda + \Delta_A + \Delta_B + m'\Delta_0)$ is a boundary set of Λ. In the limit $\Lambda \to \infty$ the right-hand side of the previous inequality vanishes. ∎

Theorem 4.4. Let A and B be self-adjoint local elements of $\mathscr{A}$. Let $k_0 > 0$. Let ω be a space translation invariant state on $\mathscr{A}$. There exists $t_0 > 0$ such that for $t \in [-t_0, t_0]$:

1. The thermodynamic limit of the linear response function $\phi_\Lambda{}^\omega(A, B; k; t)$ converges uniformly in k and t.

2. There exist elements $\xi(A, B, k; t)$ of $\mathfrak{A}$ independent of ω for which

$$\lim_{\Lambda\to\infty} \phi_\Lambda{}^\omega(A, B; k; t) = \omega(\xi(A, B; k; t))$$

3. $\omega(\xi(A, B; k; t)$ extends to a holomorphic function on $|t| \leqslant t_0$ and $|\operatorname{Im} k_\alpha| \leqslant k_0$.

Proof. One has

$$\begin{aligned}
\phi_\Lambda{}^\omega(A, B; k; t) &= \omega([A(\Lambda; k), \alpha^{\Lambda}_{-t}B(\Lambda; -k)]) \\
&= \frac{1}{N(\Lambda)} \sum_{x,y\in\Lambda} e^{ik(x-y)}\omega([\tau_x A, \alpha^{\Lambda}_{-t}\tau_y B]) \\
&= \frac{1}{N(\Lambda)} \sum_{y\in\Lambda} \sum_{x\in\Lambda-y} e^{ikx}\omega([\tau_x A, \alpha^{\Lambda-y}_{-t} B])
\end{aligned}$$

which is of the form

$$\frac{1}{N(\Lambda)} \sum_{y\in\Lambda} \mu(\Lambda - y)$$

with

$$\mu(\Lambda) = \sum_{x\in\Lambda} e^{ikx}\omega([\tau_x A, \alpha^{\Lambda}_{-t} B])$$

One now has

$$\mu(\Lambda) - \mu(\Lambda - y) = \lim_{m\to\infty} [C^{\Lambda}_{0,m}(k, t) - C^{\Lambda-y}_{0,m}(k, t)]$$

uniformly in k, t, Λ, and y (see proof of Lemma 4.1) and from the previous lemma one knows that

$$\lim_{\Lambda\to\infty} \frac{1}{N(\Lambda)} \sum_{y\in\Lambda} [C^{\Lambda}_{0,m}(k, t) - C^{\Lambda-y}_{0,m}(k, t)]$$

converges in norm to zero uniformly in k and t. Therefore the limits $\Lambda \to \infty$ and $m \to \infty$ may be interchanged. It follows that

$$\lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \sum_{y \in \Lambda} [\mu(\Lambda) - \mu(\Lambda - y)]$$

converges to zero in norm uniformly in k and t. Or

$$\lim_{\Lambda \to \infty} \left[\mu(\Lambda) - \frac{1}{N(\Lambda)} \sum_{y \in \Lambda} \mu(\Lambda - y)\right] = 0$$

But it follows from the first lemma of this section that $\lim_{\Lambda \to \infty} \mu(\Lambda)$ converges to $\omega(\xi(A, B; k; t))$ uniformly in k and t. This proves point 1 of the theorem. Points 2 and 3 now follow from Lemma 4.1 and corollary 4.2. ∎

The results of Theorems 3.2 and 4.4 may now be combined and yield:

Theorem 4.5. Under the conditions of Theorem 4.4 and with the above notations the following holds: The thermodynamic limit of the relaxation function $\Phi_\Lambda{}^\omega(A, B; k; t)$ converges uniformly in k and t, and is given by

$$\lim_{\Lambda \to \infty} \Phi_\Lambda{}^\omega(A, B; k; t) = -i \int_0^t ds \lim_{\Lambda \to \infty} \phi_\Lambda{}^\omega(B, A; -k; t)$$

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